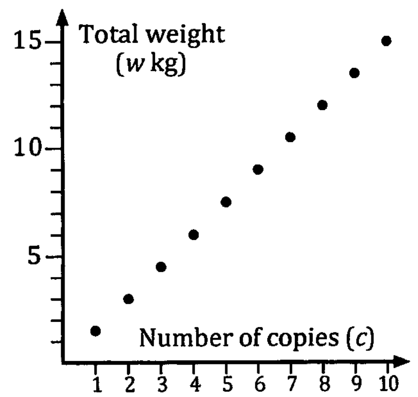
Chapter Eight.
Linear relationships.

Situation One.

Suppose that each copy of a particular book weighs 1·5 kg. If we place copies of this book on a set of scales, each time we add one more book so the weight shown will increase by 1·5 kg, as shown in the graph on the right.



Copy and complete the following table for the situation:

Number of copies, *c*.	1	2	3	4	5	6	7	8	9	10
Number of kg, *w*.										

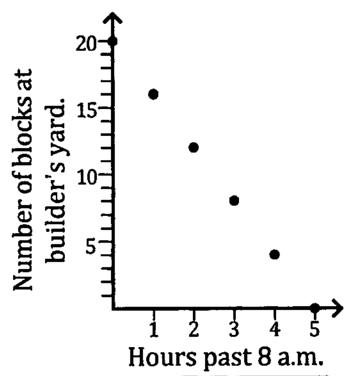
Which of the following rules agree with the figures in your table?

$w = 2c$ $c = w$ $w = 1{\cdot}5\,c$

$c = 1{\cdot}5\,w$ $c + w = 1{\cdot}5$

Situation Two.

At 8 a.m. one morning there are 20 large concrete blocks in a builders yard that need to be delivered to a worksite. A truck from a transport company is due to arrive at the builders yard in one hour to pick up four blocks, take them to the worksite and then return for four more one hour later, repeating the process until all twenty have been removed from the builders yard. Check that you agree that the graph on the right is consistent with this information.



Copy and complete the following table for this situation:

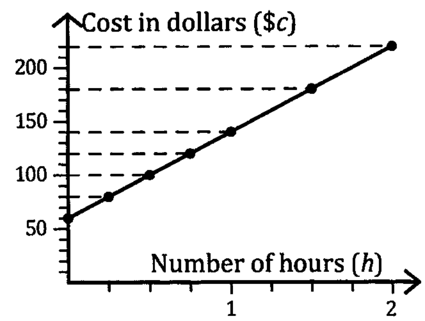
Number of hours past 8 a.m, *h*.	0	1	2	3	4	5
Number of blocks at builders yard, *n*.						

Which of the following rules agree with the figures in your table?

$n = 20h$ $n = -4h + 20$ $n = 4h - 20$ $n = 24h$ $n = 20 - h$

Situation Three.

The graph on the right shows the amount charged by a plumber working at your house for up to 2 hrs.



Copy and complete the following table for this situation:

Number of hours, *h*.	0	0·25	0·5	0·75	1	1·5	2
Cost in dollars, *c*.							

Which of the following rules agree with your table?

$c = 80 + 60h$ $c = 80h + 60$ $c = 60 - 80h$

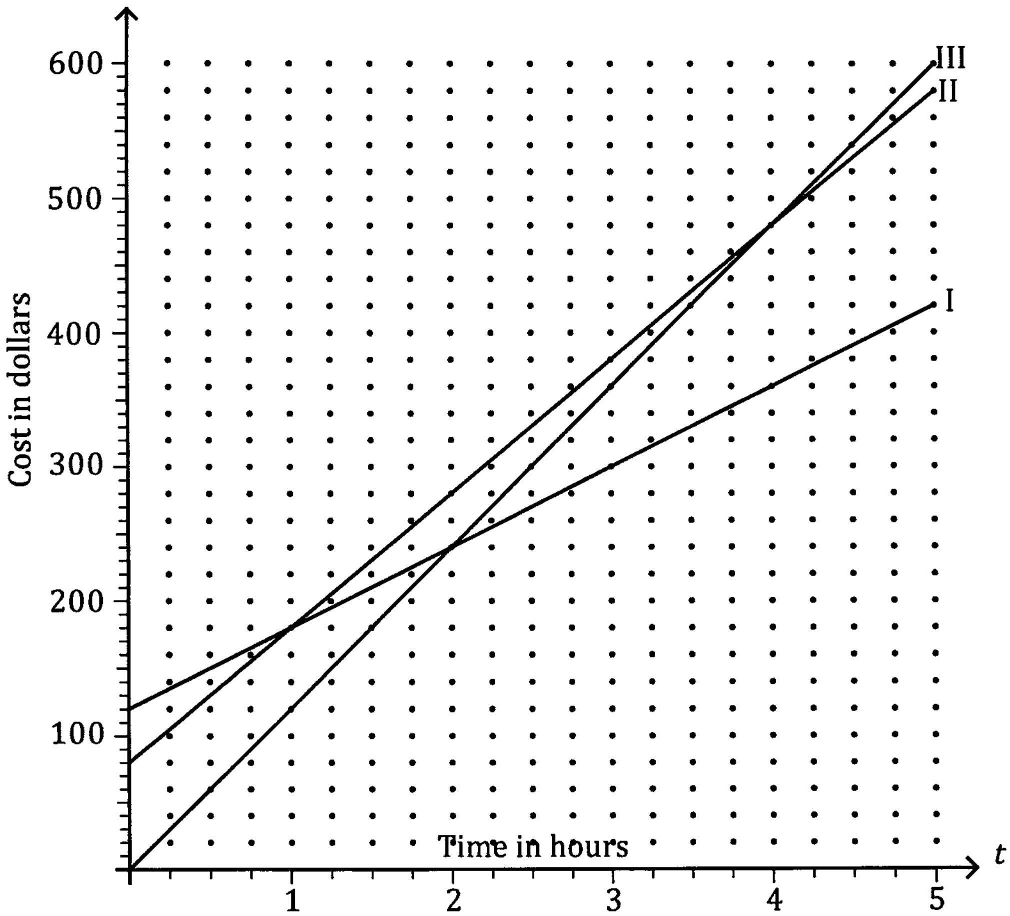
Why are the dots joined in this graph and not for situations one and two?

Situation Four.

Three electricians, Sparky, Flash and Voltman, have different ways of calculating a customer's bill.

- Sparky charges a standard rate per hour and has no other charges.
- Flash has a fixed, or "standing" charge and then charges a certain amount per hour on top of that.
- Voltman has a higher standing charge than Flash but then charges less per hour.

These three methods are shown graphed below:

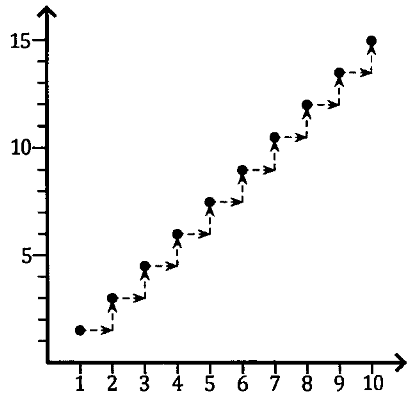


- Which line, I, II or III, corresponds to (a) Sparky, (b) Flash, (c) Voltman?
- Ignoring the standing charges who charges most per hour?
  What feature of the graph shows this?
- With the charge, or cost, being \$C and the time being t hours the equation of line I is
  $$C = 60t + 120.$$
  Determine the equations of lines II and III.
- If you were considering using one of the three electricians for a job and wanted to keep the cost to a minimum which of the three could you dismiss from your considerations?

Straight line graphs.

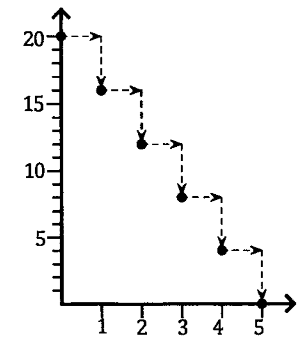
The situations on the previous two pages each gave rise to graphs for which the plotted points lay in a straight line. This is because in each situation, for each unit increase horizontally the vertical change remains constant. For example, for the first three situations:

Situation one.



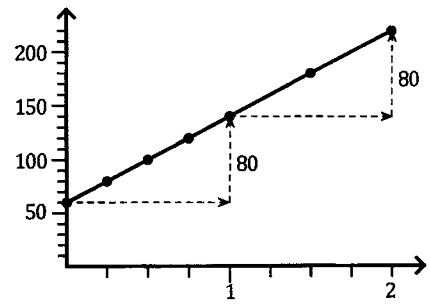
Each time we move right 1 unit we move up 1·5 units.

Situation two.



Each time we move right 1 unit we move down 4 units

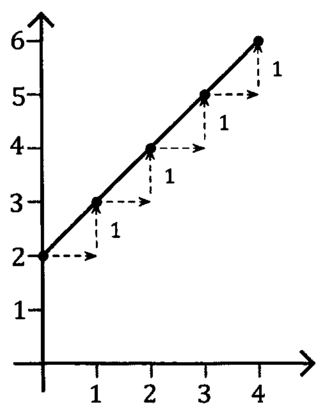
Situation three.



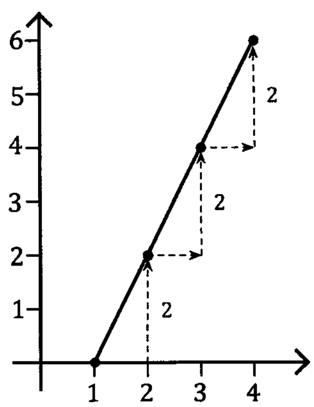
Each time we move right 1 unit we move up 80 units. (Moving right one quarter of a unit sees a vertical rise of just 20 units.)

The gradient of a straight line.

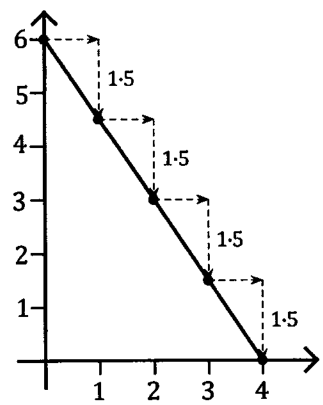
This vertical rise for each horizontal unit increase is called the **gradient** or **slope** of the straight line. For example:



Each time we move right 1 unit we move up 1 unit.
Gradient = 1



Each time we move right 1 unit we move up 2 units.
Gradient = 2

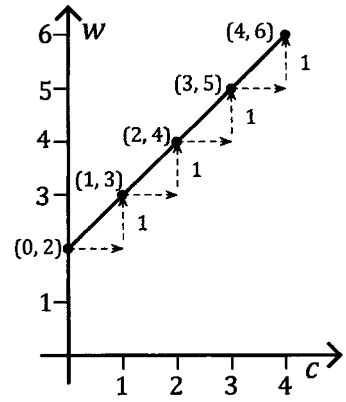


Each time we move right 1 unit we move **down** 1·5 units.
Gradient = −1·5

Note carefully the use of the negative gradient in the third case to indicate that the line moves **down** as we move to the right.

Table of values.

Using points with whole number coordinates to create a table of values for the three graphs just encountered, with c as the horizontal coordinate and w the vertical coordinate, the table for the first graph (shown again on the right) is:



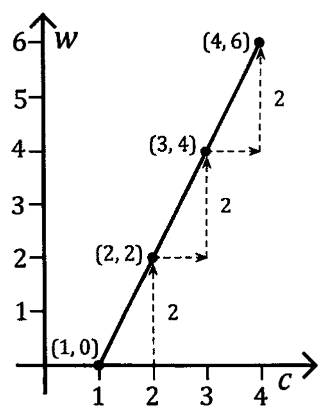
1 1 1 1

c	0	1	2	3	4
w	2	3	4	5	6

1 1 1 1

Notice how the gradient of the line, i.e. the constant increase in the w-values for each unit increase in the c-values, is evident from the table.

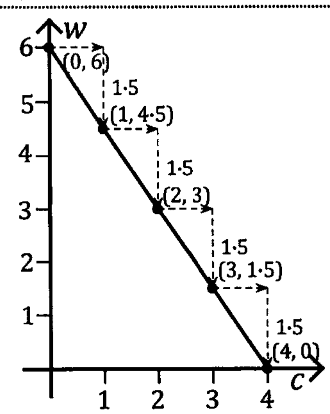
For the other two graphs the tables (and graphs) are shown below:



1 1 1

c	1	2	3	4
w	0	2	4	6

2 2 2



1 1 1 1

c	0	1	2	3	4
w	6	4·5	3	1·5	0

−1·5 −1·5 −1·5 −1·5

Notice again how the gradient of 2 and of −1·5 are evident from these tables.

☛ **The gradient of a straight line graph is the vertical rise in the graph for each unit moved to the right. (A fall in the graph for each unit moved right indicates that the gradient is negative.)**

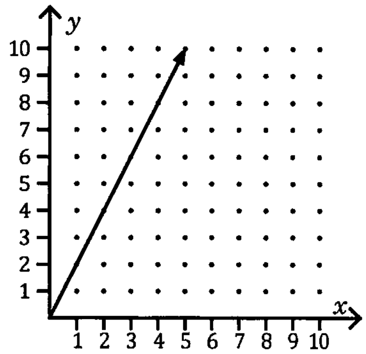
This gradient, or *slope*, is sometimes described as "*the rise divided by the run*".

☛ **With the horizontal coordinate increasing by 1 unit each time, the table of values for a straight line graph will show a constant difference pattern in the vertical coordinate equal to the gradient of the line. (If, for a constant increase in the values of the horizontal coordinate, the values of the vertical coordinate do not show a constant difference pattern the table of values is not for a straight line graph.)**

Exercise 8A

For each of questions 1 to 20 determine the gradient of the given straight line.

1.



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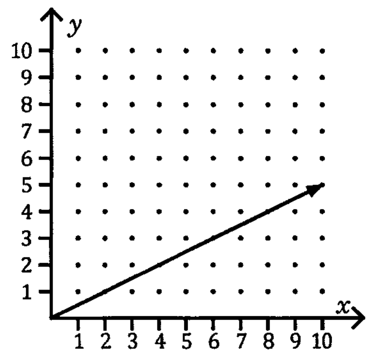

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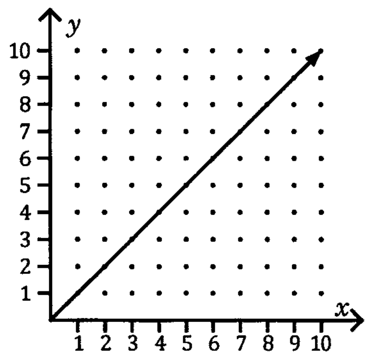

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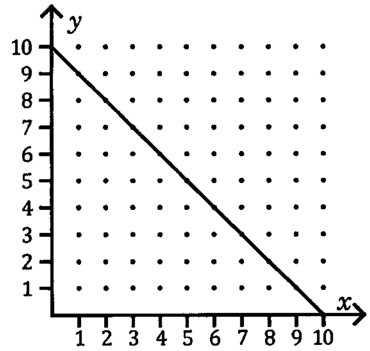

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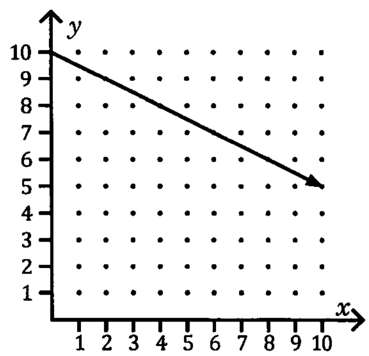
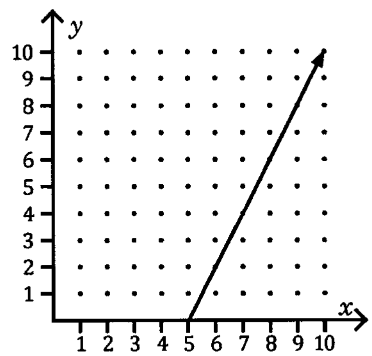


6.



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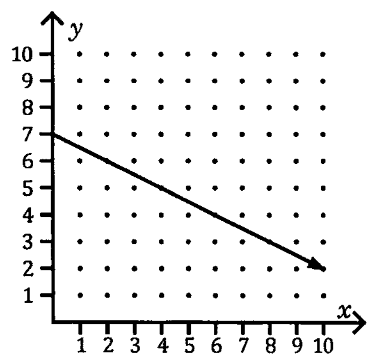

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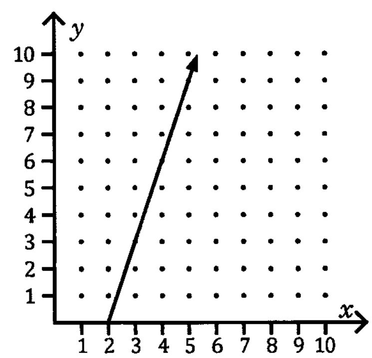

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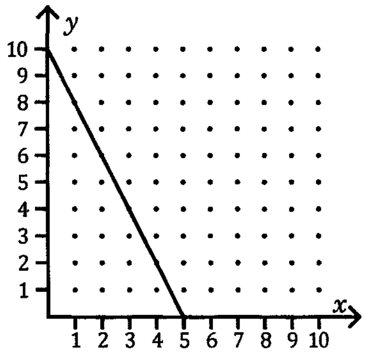
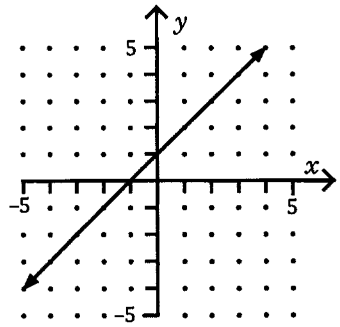


10.



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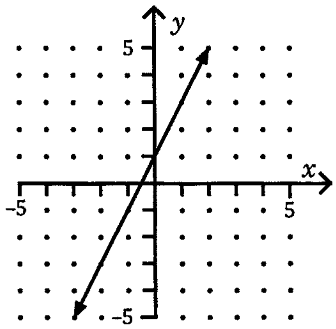

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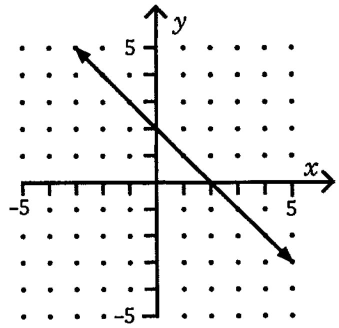

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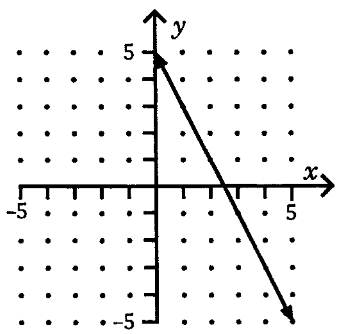

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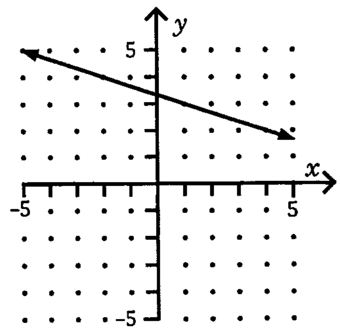

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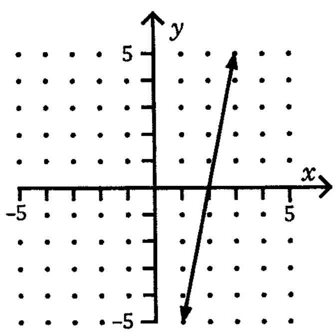

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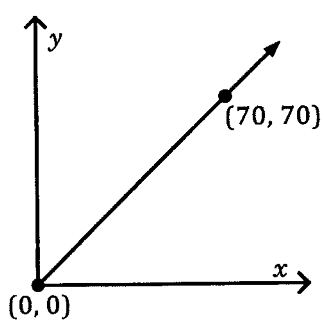

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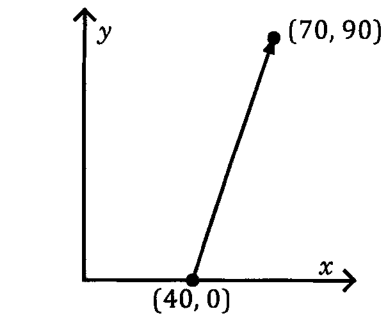

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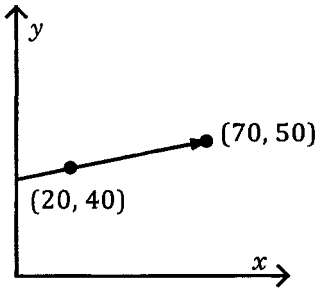

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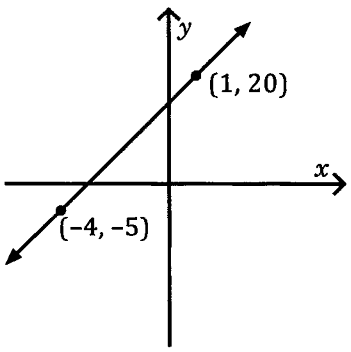
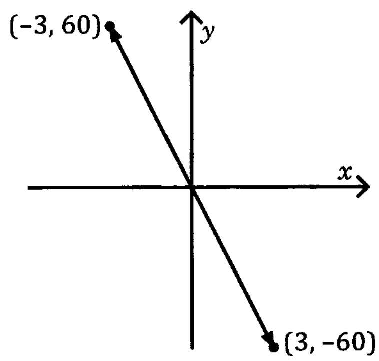


20.



For questions 21 to 34 state whether or not the given values (x, y) would lie in a straight line if plotted and, for those cases for which a straight line would be formed, what would be the gradient of that straight line?

21.

x	0	1	2	3	4	5
y	4	6	8	10	12	14

22.

x	0	1	2	3	4	5
y	19	17	15	13	11	9

23.

x	0	1	2	3	4	5
y	0	1	4	9	16	25

24.

x	6	7	8	9	10	11
y	5	7	1	3	16	10

25.

x	7	8	9	10	11	12
y	12	17	22	27	32	37

26.

x	0	2	4	6	8	10
y	20	19	17	14	10	5

27.

x	1	3	5	7	9	11
y	50	40	30	20	10	0

28.

x	3	4	6	9	13	16
y	3	5	7	9	11	13

29.

x	6	5	4	3	2	1
y	12	15	18	21	24	27

30.

x	2	3	4	5	6	7
y	−17	−14	−11	−8	−5	−2

31.

x	1	3	4	5	6	7
y	8	12	14	16	18	20

32.

x	4	2	6	1	3	5
y	17	11	23	8	14	20

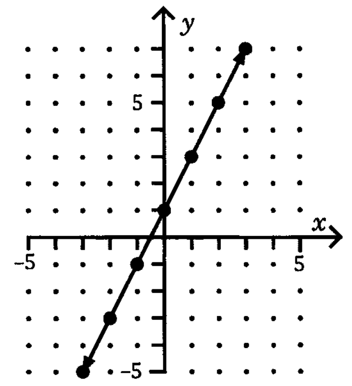
33.

x	11	13	9	12	10	8
y	24	48	8	35	15	3

34.

x	8	14	4	10	6	12
y	11	26	1	16	6	21

What's my rule?



Notice that the straight line graph on the right passes through the following points (as well as others):

(-3, -5), (-2, -3), (-1, -1), (0, 1),
(1, 3), (2, 5), (3, 7).

For every one of these points the x and y coordinates fit the rule $y = 2x + 1$.

Indeed every point lying on the given straight line will have coordinates that fit the rule

$$y = 2x + 1$$

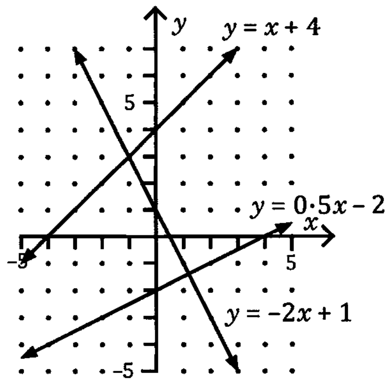
and all points not on the line will not fit the rule.

For example: The point (1·5, 4) lies on the line and
$4 = 2 \times 1{\cdot}5 + 1$
The point (−2·5, −4) lies on the line and
$-4 = 2 \times (-2{\cdot}5) + 1$
The point (2, 3) does not lie on the line and
$3 \neq 2 \times 2 + 1$
The point (−3, 0) does not lie on the line and
$0 \neq 2 \times (-3) + 1$

We say that the rule for the straight line shown is $y = 2x + 1$.

Note that the line with a rule of $y = 2x + 1$ has
- a gradient of 2

and
- cuts the vertical axis at (0, 1).



Similarly: $y = 1x + 4$ has
- a gradient of 1

and
- cuts the y-axis at (0, 4).

$y = 0{\cdot}5x - 2$ has
- a gradient of 0·5

and
- cuts the y-axis at (0, −2).

$y = -2x + 1$ has
- a gradient of −2

and
- cuts the y-axis at (0, 1).

To generalise:

The straight line with the rule $y = mx + c$ **has** • **a gradient of m** **and** • **cuts the y-axis at the point (0, c).**

And:

If a straight line has gradient m and cuts the y-axis at the point (0, c) it has the rule: $y = mx + c$

Note: In saying that the rule for a straight line has the form y = mx + c the choice of the letters m and c is not important. We could equally well have said that straight lines have rules of the form

y = px + k, y = ax + b, y = bx + a, y = cx + d, y = rx + s, etc.

or as

y = k + px, y = b + ax, y = a + bx, y = d + cx, y = s + rx, etc.

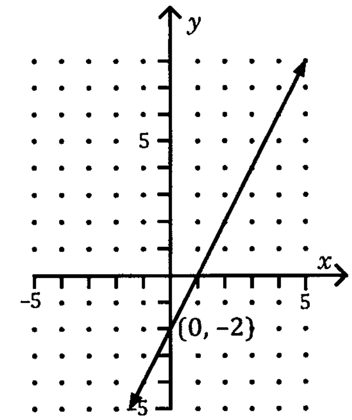
It is the form of the rule that is important, not the use of m and c.
However y = mx + c is one of the expressions more frequently used, as is y = a + bx, so in this text we will tend to use one or other of these two.

For the straight line shown on the right:

Gradient = 2 (Each move of one unit to the right sees the line go up 2 units.)

The line cuts the vertical axis at (0, -2).

Thus the rule for the line is $y = 2x - 2$.



The points (2, 2) and (4, 6) lie on the line. Confirm that these values for x and y do indeed fit the rule.

Choose another point on the line and similarly confirm that its coordinates fit the rule.

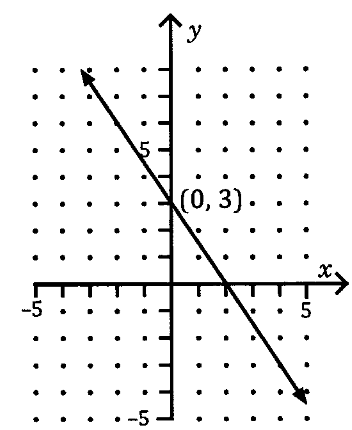
Choose a point that does not lie on the line and confirm that its coordinates do not fit the rule.

For the straight line shown on the right:

Gradient = –1·5 (Each move of one unit to the right sees the line go down 1·5 units.)

The line cuts the vertical axis at (0, 3).

Thus the rule for the line is $y = -1{\cdot}5x + 3$.



The points (2, 0) and (–2, 6) lie on the line. Confirm that these values for x and y do indeed fit the rule.

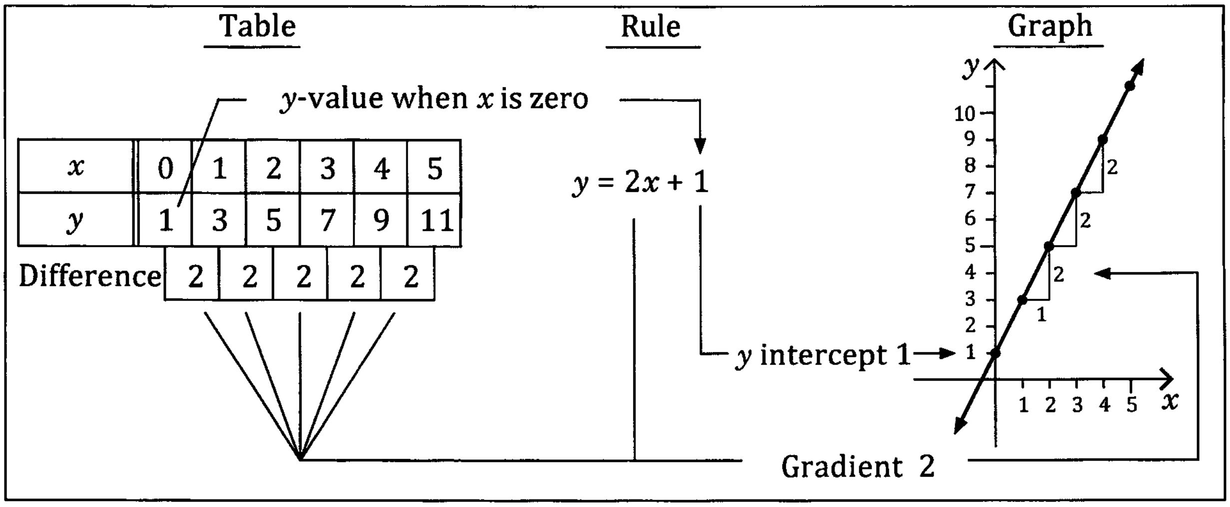
Choose another point on the line and similarly confirm that its coordinates fit the rule.

Choose a point that does not lie on the line and confirm that its coordinates do not fit the rule.

Table – Rule – Graph.

If the x and y values are linked by a rule of the form y = mx + c then:

- ☞ A straight line, or **linear**, relationship exists between x and y.
- ☞ Graphing, with the x values on the horizontal axis and the y values on the vertical axis, produces a straight line with gradient m and cutting the vertical axis at the point with coordinates (0, c).
  We call c the **vertical intercept**, or y-intercept.
- ☞ With the x values increasing by 1 the **common difference** in the y values is equal to m, the gradient of the line.



Example 1

For each of the following tables determine whether the relationship between the two variables is linear and, for any that are, determine the rule.

(a)

x	0	1	2	3	4	5
y	−2	1	4	7	10	13

(b)

P	3	4	5	6	7	8
t	8	15	24	35	48	63

(c)

r	2	4	1	5	3	6
s	17	31	10	38	24	45

(a)

x	0	1	2	3	4	5
y	−2	1	4	7	10	13
Difference	3	3	3	3	3	

Constant difference pattern thus relationship is linear.
The rule will be of the form

$$y = 3x + c.$$

To fit the tabulated data the rule must be $y = 3x - 2$.

(b)

P	3	4	5	6	7	8
t	8	15	24	35	48	63
Difference	7	9	11	13	15	

Difference pattern is not constant thus relationship is not linear.

(c) First present r values in order:

r	1	2	3	4	5	6
s	10	17	24	31	38	45
Difference	7	7	7	7	7	

Constant difference pattern thus relationship is linear.
The rule will be of the form

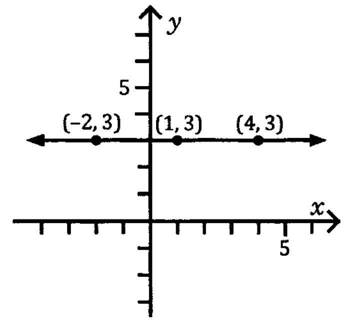
$$s = 7r + c.$$

To fit the tabulated data the rule must be $s = 7r + 3$.

Lines parallel to the axes.

I. Lines parallel to the x-axis.

Consider the line parallel to the x-axis and passing through (−2, 3), (1, 3) and (4, 3) as shown on the right.



The rule for this line is $y = 3$ because the y coordinates of all points lying on this line will equal 3.

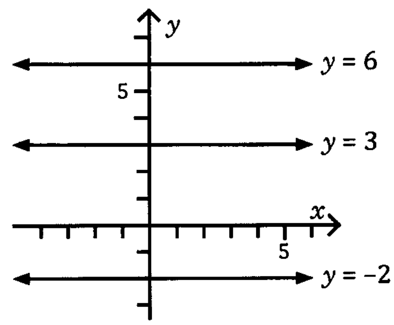
Note that this is consistent with the $y = mx + c$ idea because the line has a gradient of zero and a y intercept of 3.

The graph shown on the right shows the horizontal lines:

$y = 6$,

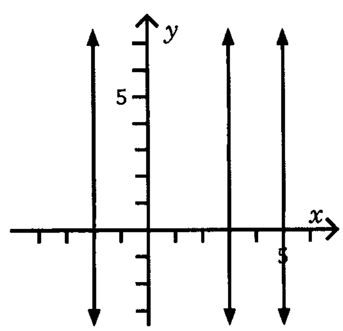
$y = 3$,

and $y = -2$.



II. Lines parallel to the y-axis.

The diagram on the right shows some lines drawn parallel to the y-axis.



The gradient of each of these vertical lines is undefined because we cannot find the vertical rise in the line for each horizontal unit increased because the line rises vertically for zero increase horizontally! Hence we should not expect the rules for vertical lines to be of the form $y = mx + c$ because the gradient, m, is undefined. Indeed straight lines parallel to the y-axis are the only straight lines having rules that are <u>not</u> of the form $y = mx + c$.

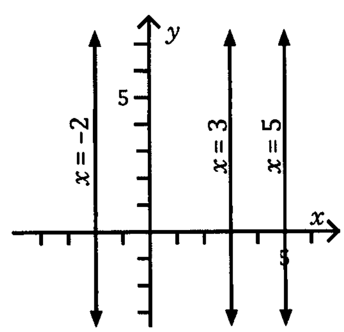
Lines parallel to the y-axis have rules of the form $x = c$.

The graph on the right shows the vertical lines:

$x = -2$

$x = 3$,

and $x = 5$.

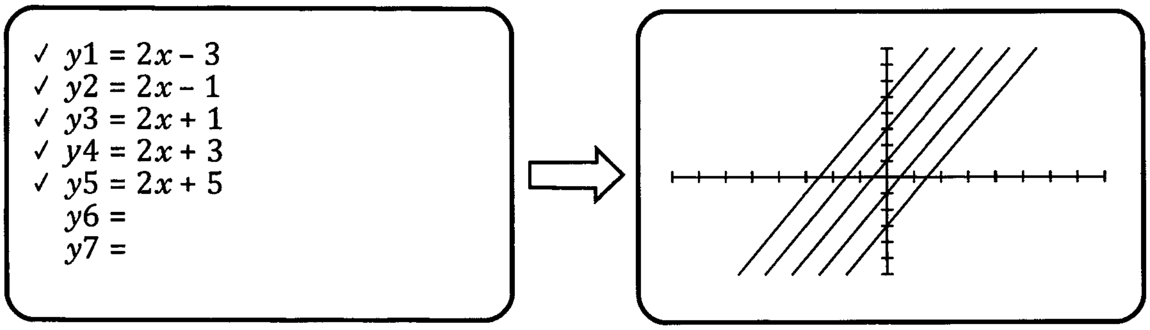


Even though these vertical lines have rules of a different form, points lying on each line must still "obey" the rule. For example, for a point to lie on $x = 3$ the point must have an x-coordinate equal to 3.

Use of a calculator with a graphing facility.
Calculators with a graphing facility can display graphs of lines given the rules for the lines. For example, entering the following rules into such a calculator

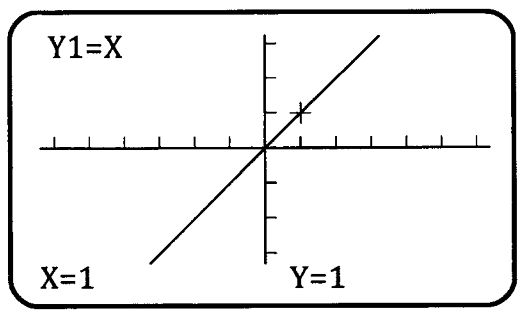
$y = 2x - 3$, $y = 2x - 1$, $y = 2x + 1$, $y = 2x + 3$ and $y = 2x + 5$

allows the graphs of these lines to be displayed.

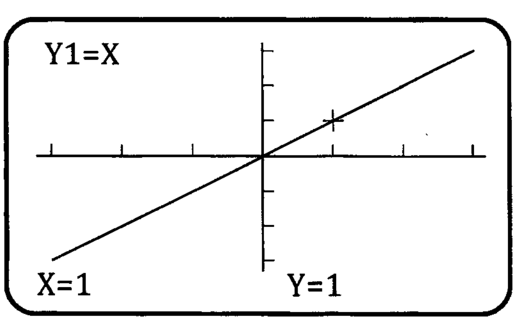


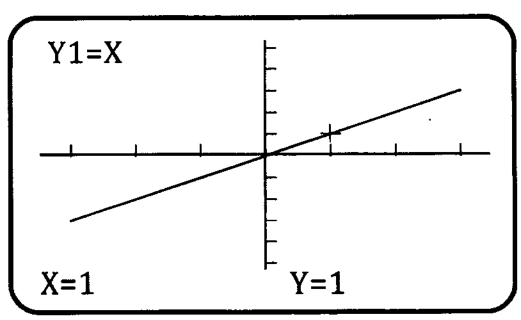
What feature of the five rules allows us to anticipate that the lines would be parallel to each other?

Note: The display on the right shows the line $y = x$. As we would expect this line has a gradient of 1 and cuts the y-axis at the point (0, 0), i.e. the origin. The line passes through all those points for which the x-coordinate equals the y-coordinate, for example (0, 0), (1, 1), (2, 2) etc.



Because the same scale is used on each axis the line $y = x$ makes an angle of 45° with each axis. However, do not expect this 45° property of the line $y = x$ to be evident if different scales are used on each axis. Both of the displays shown below show the line $y = x$ but with different scales used on each axis, and with the two displays using different scales, the two displays appear different and neither shows the 45° nature of the line.





Exercise 8B

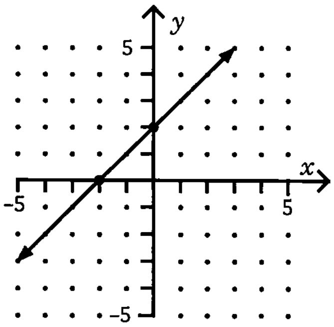
For each of questions 1 to 10 determine:

(a) the gradient of the line,

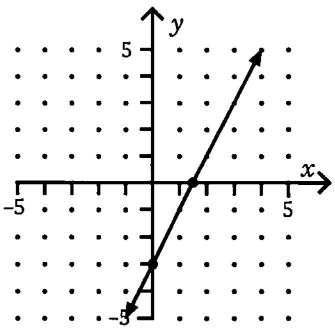
(b) the coordinates of the point where the line cuts the vertical axis,

(c) the rule for the line.

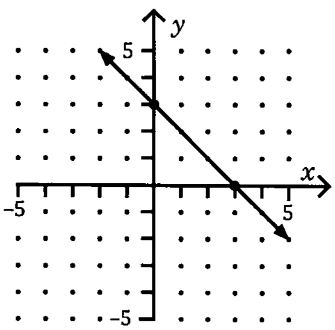
1.



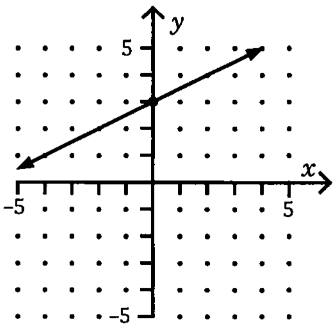
2.



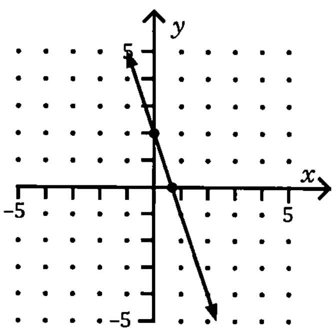
3.



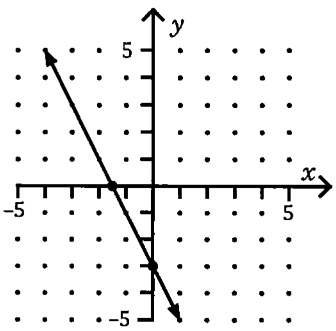
4.



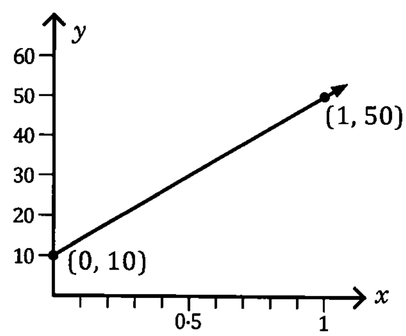
5.



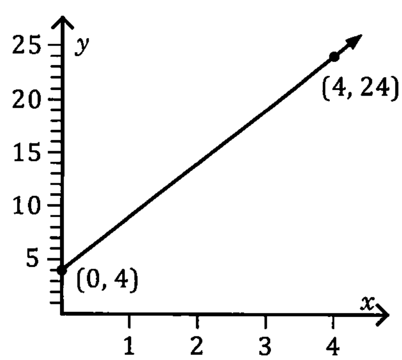
6.



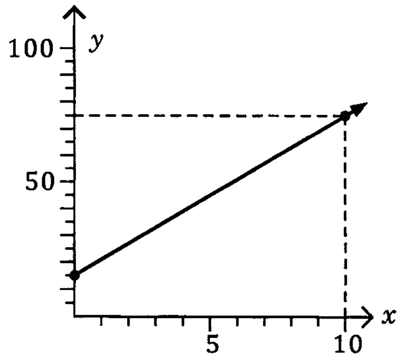
7.



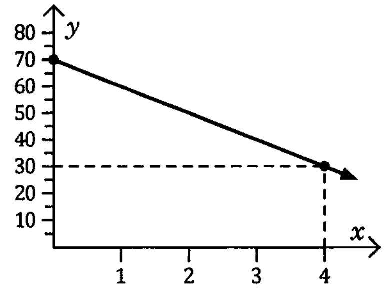
8.



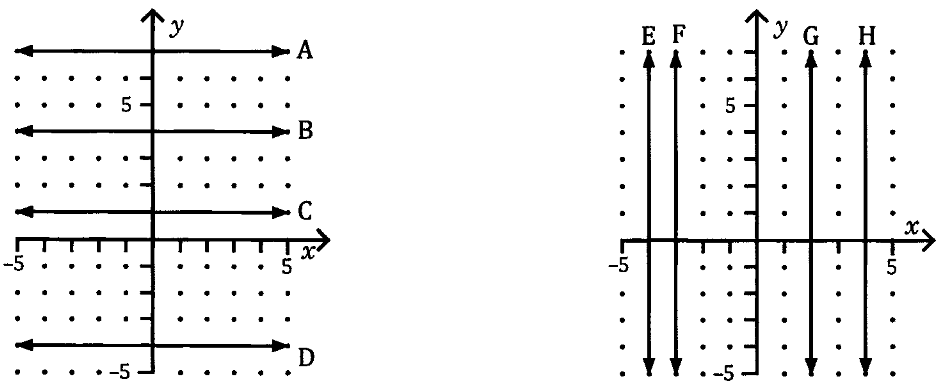
9.



10.



11. Determine the rules for each of the lines A to H shown below.



For each of the tables shown in questions 12 to 25 determine whether the relationship between x and y is linear and, for any for which it is, determine the rule for the relationship.

12.

x	1	2	3	4	5	6
y	4	7	10	13	16	19

13.

x	1	2	3	4	5	6
y	21	17	13	9	5	1

14.

x	0	1	2	3	4	5
y	-3	2	7	12	17	22

15.

x	0	1	2	3	4	5
y	25	16	9	4	1	0

16.

x	1	2	3	4	5	6
y	2	3·5	5	6·5	8	9·5

17.

x	5	6	7	8	9	10
y	5	10	20	40	80	160

18.

x	-2	-1	0	1	2	3
y	0	1	2	3	4	5

19.

x	0	2	4	6	8	10
y	1	3	6	10	15	21

20.

x	1	3	5	7	9	11
y	12	10	8	6	4	2

21.

x	0	5	10	15	20	25
y	21	31	41	51	61	71

22.

x	2	5	1	6	4	3
y	10	19	7	22	16	13

23.

x	6	3	5	2	1	4
y	18	3	13	-2	-7	8

24.

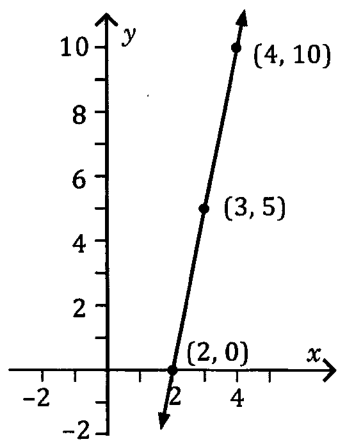
x	5	13	9	3	11	7
y	8	24	16	4	20	12

25.

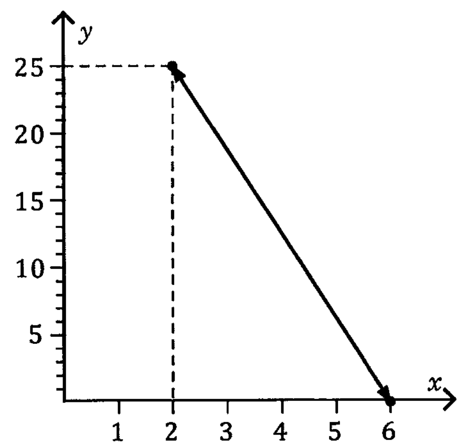
x	16	7	1	13	4	10
y	49	22	4	40	13	31

For 26 and 27 find (a) the gradient of the line,
(b) the coordinates of the point where the line will cut the y - axis,
(c) the rule for the line.

26.



27.



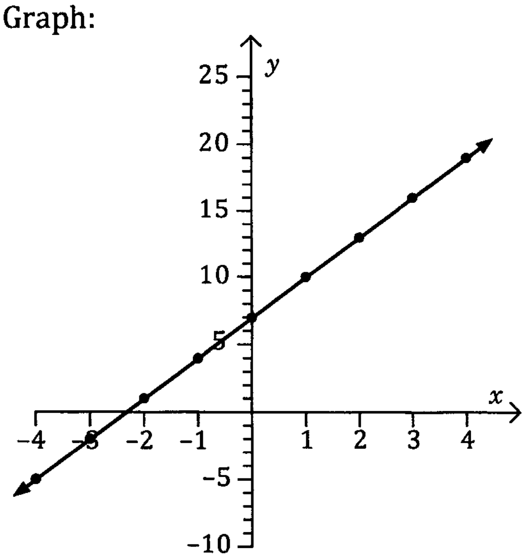
In each of questions 28 to 33 you are given a table of values, OR a rule OR a graph. Use the one you are given to complete the other two. (When drawing the graph assume all x values are possible, not just the integer values given in the table.)

28. Table:

x	-4	-3	-2	-1	0	1	2	3	4
y									

Rule: $y =$

Graph:

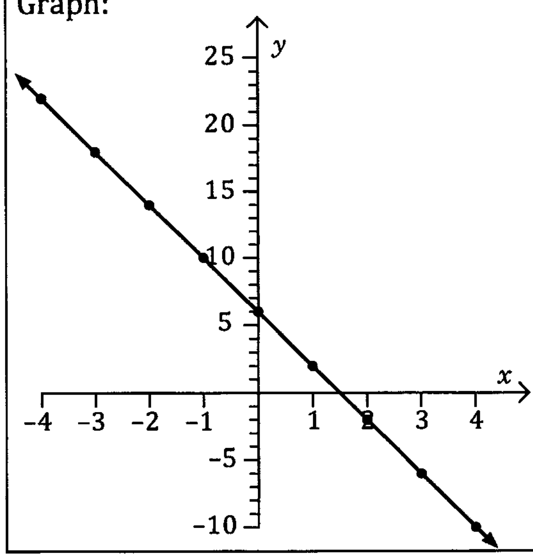


29. Table:

x	-4	-3	-2	-1	0	1	2	3	4
y									

Rule: $y =$

Graph:

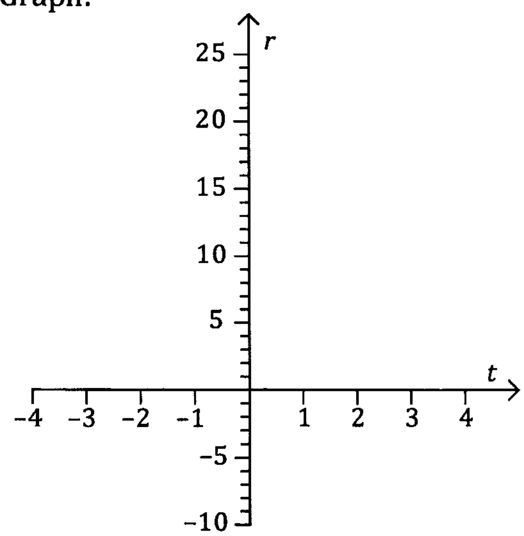


30. Table:

t	-4	-3	-2	-1	0	1	2	3	4
r	-2	0	2	4	6	8	10	12	14

Rule: r =

Graph:

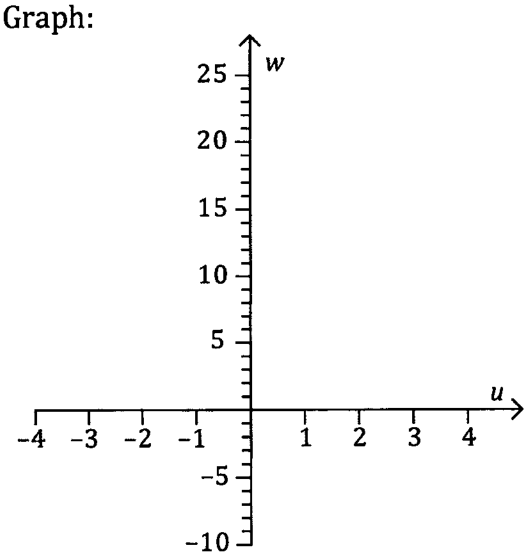


31. Table:

u	-4	-3	-2	-1	0	1	2	3	4
w	-9	-6	-3	0	3	6	9	12	15

Rule: w =

Graph:

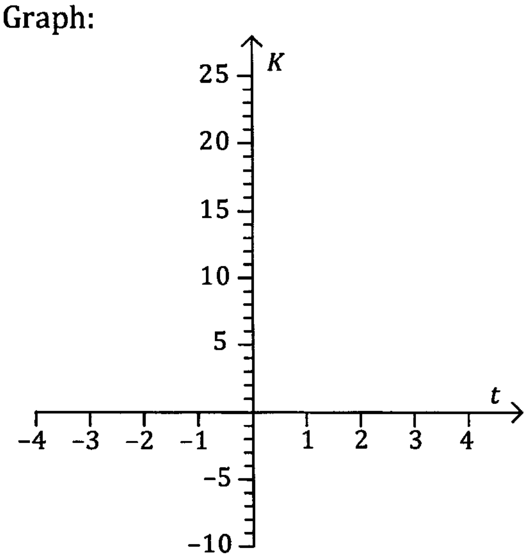


32. Table:

t	-4	-3	-2	-1	0	1	2	3	4
K									

Rule: $K = 4t + 9$

Graph:

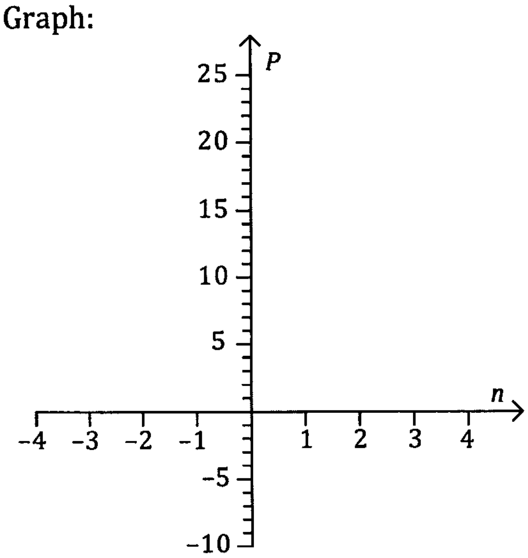


33. Table:

n	-4	-3	-2	-1	0	1	2	3	4
P									

Rule: $P = -2n + 7$

Graph:



34. On squared paper, and with each axis from −6 to 6, use gradients and intercepts to sketch the following four lines on the one graph:
$y = 2x + 3$ $\quad y = -2x + 1$ $\quad y = -0{\cdot}5x + 4$ $\quad y = 0{\cdot}5x - 4$
Now use your graphic calculator to check the correctness of your sketch.

35. The display on the right shows the lines

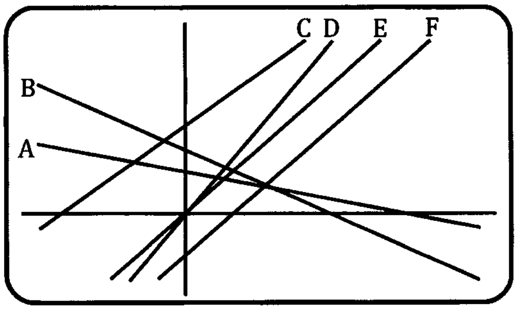
$y = 3x$ $\quad y = 4x$

$y = 3x - 2$ $\quad y = 2{\cdot}5x + 4$

$y = -1{\cdot}5x + 3$ $\quad y = -\frac{2}{3}x + 2$

labelled A to F.

Allocate the correct rule to each line.



36. The display on the right shows the lines

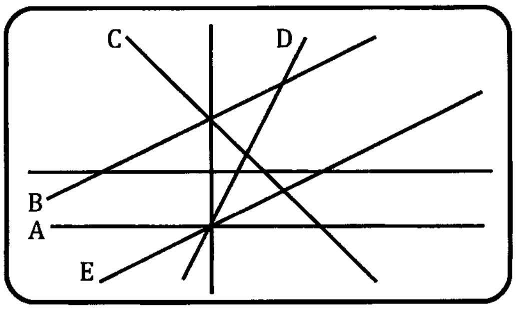
$y = 0{\cdot}5x + 2$ $\quad y = 0{\cdot}5x - 2$

$y = -2$ $\quad y = -x + 2$

$y = 2x - 2$

labelled A to E.

Allocate the correct rule to each line.



37. The display on the right shows the lines

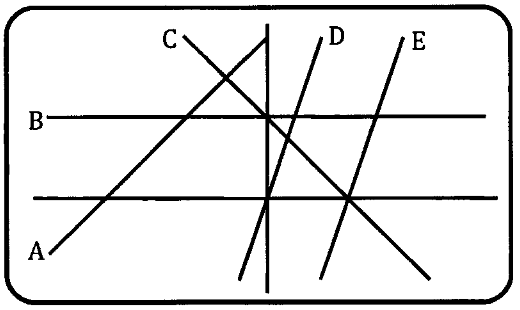
$y = 30x$ $\quad y = 30$

$y = 30x - 90$ $\quad y = 10x + 60$

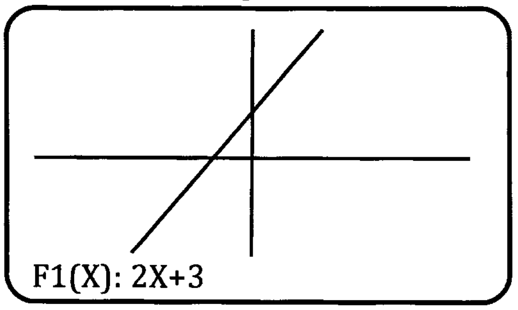
$y = -10x + 30$

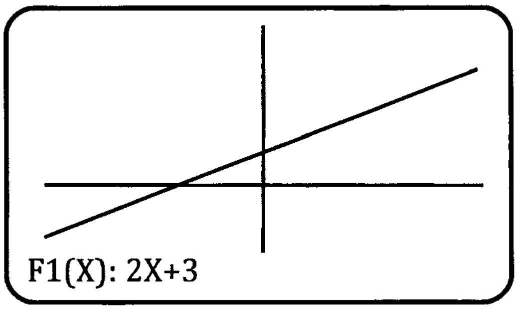
labelled A to E.

Allocate the correct rule to each line.



38. Both of the displays shown below show the line $y = 2x + 3$. How is it that they can look different? Explain.





More about $y = mx + c$, the equation of a straight line.
From earlier work we know that straight lines have equations that are of the form:

$$y = mx + c$$

① Why should equations of the form $y = mx + c$ produce straight line graphs?
A straight line must clearly have a constant gradient. Thus wherever we are on the line, if we journey one further unit horizontally, then the line will always rise a constant vertical amount.
Consider the equation $y = mx + c$.
If x changes from 2 to 3, y changes from $(2m + c)$ to $(3m + c)$, a rise of m.
If x changes from 3 to 4, y changes from $(3m + c)$ to $(4m + c)$, a rise of m.
If x changes from 98 to 99, y changes from $(98m + c)$ to $(99m + c)$, a rise of m.
Each unit increase in the x-coordinate produces an increase of m units in the y-coordinate. i.e. a straight line of gradient m.
If an equation cannot be expressed in the form $y = mx + c$ then the y-coordinate will not rise by a constant amount for each unit increase in the x-coordinate.
For example, consider $y = x^2$.
If x changes from 2 to 3, y changes from 4 to 9 i.e. a rise of 5.
If x changes from 3 to 4, y changes from 9 to 16 i.e. a rise of 7.
If x changes from 98 to 99, y changes from 9604 to 9801 i.e. a rise of 197.
Each increase of one unit in the x-coordinate does not produce the same change in the y-coordinate. i.e. not a straight line graph.

② The equation of a line is like the membership ticket for the line.
As was mentioned earlier, if a point lies on a particular line then the coordinates of that point will "fit" the equation of the line and if it does not lie on the line the coordinates will not "fit" the equation. The equation of a line is the rule which all points lying on the line must "obey". In this way the equation is like the membership ticket for the line. If a point does not fit the equation it cannot lie on the line.

Example 2
State whether each of the points A to C lie on the line $y = 2x + 7$.
A (1, 8) B(3, 13) C (−5, −3)

If A (1, 8) lies on the line then $x = 1$ and $y = 8$ must "fit" the equation.
Substituting $x = 1$ into $y = 2x + 7$ gives $y = 9$.
Thus point A (1, 8) does not lie on the line $y = 2x + 7$.

If B (3, 13) lies on the line then $x = 3$ and $y = 13$ must "fit" the equation.
Substituting $x = 3$ into $y = 2x + 7$ gives $y = 13$.
Thus point B (3, 13) does lie on the line $y = 2x + 7$.

If C (−5, −3) lies on the line then $x = -5$ and $y = -3$ must "fit" the equation.
Substituting $x = -5$ into $y = 2x + 7$ gives $y = -3$.
Thus point C (−5, −3) does lie on the line $y = 2x + 7$.

It may not look like $y = mx + c$ but it may still be linear.
Straight lines have equations that can be written in the form $y = mx + c$.

Consider each of the following equations:

$$2y + 3x = 12$$
$$5x - 2y = 15$$
$$\frac{y}{3} = \frac{1}{2} - \frac{2x}{5}$$

Each of these can be rearranged into the form $y = mx + c$:

$$\begin{aligned} 2y + 3x &= 12 \\ 2y &= -3x + 12 \\ y &= -1{\cdot}5x + 6 \end{aligned}$$

$$\begin{aligned} 5x - 2y &= 15 \\ 5x - 15 &= 2y \\ y &= 2{\cdot}5x - 7{\cdot}5 \end{aligned}$$

$$\begin{aligned} \frac{y}{3} &= \frac{1}{2} - \frac{2x}{5} \\ \times \text{ by } 3 \quad y &= -\frac{6x}{5} + \frac{3}{2} \end{aligned}$$

Hence whilst they may not initially look like $y = mx + c$ each of the three equations are equations of straight lines.

Determining the equation of a straight line.
We can determine the equation if we know:

- ☛ the gradient and the vertical intercept (see example 3 that follows)

or ☛ the gradient and one point on the line (see example 4)

or ☛ if we know two points on the line (see example 5).

Example 3 (Given the gradient and the vertical intercept.)
State the equation of the straight line that cuts the y-axis at the point (0, 1) and has a gradient of 6.

A line with gradient m and cutting the y-axis at (0, c) has equation $y = mx + c$.
Thus the given line has equation $y = 6x + 1$.

Example 4 (Given the gradient and one point on the line.)
Find the equation of the straight line through the point (4, −3) and with a gradient of −2.

A straight line of gradient m has an equation of the form $y = mx + c$.
Thus the given line will have an equation of the form $y = -2x + c$.
The line passes through the point (4, −3).
Thus the values $x = 4$ and $y = -3$ must "fit" the equation, i.e. $(-3) = -2(4) + c$
giving $c = 5$.
Thus the given line has equation $y = -2x + 5$.

Example 5 (Given two points that lie on the line.)
Find the equation of the straight line through the points (−2, 8) and (4, −1).

Starting at the point with the lower x-coordinate (−2, 8), and moving to the point (4, −1), we travel across 6 units and down 9 units. Thus in moving across 1 unit we travel <u>down</u> $\frac{9}{6}$ units, i.e. $\frac{3}{2}$ units. The gradient of the line is $-\frac{3}{2}$.

Thus the given line will have an equation of the form $y = -1{\cdot}5x + c$.

The line passes through the point (4, −1).
Thus the values $x = 4$ and $y = -1$ must "fit" the equation, i.e. $-1 = -1{\cdot}5(4) + c$
giving $c = 5$.

Thus the given line has equation $y = -1{\cdot}5x + 5$.

(The reader should confirm that using the point (−2, 8) and saying that the values $x = -2$ and $y = 8$ must "fit" the equation also gives c = 5.)

A useful rule.
A useful rule to remember when determining the gradient of the line through two points, A and B is:

$$\text{Gradient} = \frac{\text{the change in the } y\text{-coordinate in going from A to B}}{\text{the change in the } x\text{-coordinate in going from A to B}}.$$

Thus if A has coordinates (x_1, y_1) and B has coordinates (x_2, y_2) then the gradient of line through A and B = $\dfrac{y_2 - y_1}{x_2 - x_1}$.

Note: In this formula $\dfrac{y_1 - y_2}{x_1 - x_2}$ would also give the correct answer but $\dfrac{y_1 - y_2}{x_2 - x_1}$ and $\dfrac{y_2 - y_1}{x_1 - x_2}$ would not. Hence make sure that "whichever point you get the first y-coordinate from is also where you get the first x-coordinate from".

Calculator routines.
Your calculator may have programmed routines that allow the equation of a line to be determined simply by inputting the coordinates of two points on the line, or inputting the gradient and the coordinates of just one point on the line. Such routines can be useful but make sure that you understand the underlying ideas and can apply them without the assistance of calculator programs if required.

Exercise 8C

1. Calculate the gradient of the straight line through each of the following pairs of points.

(a) (4, 6) and (2, 2)	(b) (6, 7) and (5, 3)	(c) (4, 5) and (2, 1)
(d) (6, 7) and (2, 5)	(e) (5, 3) and (1, 2)	(f) (5, 3) and (4, 2)
(g) (4, 3) and (2, 7)	(h) (5, 2) and (3, −3)	(i) (4, 2) and (−2, −1)
(j) (1, −7) and (−1, 1)	(k) (−1, −2) and (1, 3)	(l) (2, −3) and (6, −1)

2. State the gradient and the coordinates of the y-axis intercept of each of the following straight lines.

(a) $y = 3x - 17$	(b) $y = -2x + 13$	(c) $y = 5 - 7x$
(d) $2x + 3y = 24$	(e) $5y + 2x = 8$	(f) $2x - 3y + 9 = 0$
(g) $\frac{x}{2} + y = 11$	(h) $\frac{y}{5} + \frac{x}{2} = 3$	(i) $\frac{2x}{5} + \frac{y}{3} = 4$

3. What is the equation of the x axis?

4. What is the equation of the y axis?

5. State whether each of the points A to E lie on the line $y = 3x - 5$.
   A (6, 12) B(5, 11) C (2, 1) D (−3, −13) E (−1, −8)

6. State which of the points F to J do <u>not</u> lie on the line $y = -x + 6$.
   F (1, 5) G(0, 6) H (2, 8) I (−1, 4) J (6, 0)

7. Write down the equation of the straight line with gradient 3 and cutting the y-axis at (0, 4). Does this line pass through the point (−1, 1) ?

8. Write down the equation of the straight line cutting the y-axis at (0, 2) and with gradient 0·5. Which of the following points lie on this line ?
   A (2, 1), B (2, 0), C (4, 2), D (−6, −1), E (4, 4).

9. Given that all of the points A to F given below lie on the line $y = 2x - 5$ determine the values of a, b, c, d, e and f.
   A (3, a), B (2, b), C (−4, c), D (2·5, d), E (e, 13), F (f, −5).

10. Find the equation of each of the following straight lines.

(a) Gradient 1, through (3, 5).	(b) Gradient −1, through (6, −1).
(c) Gradient −2, through (3, 2).	(d) Gradient 5, through (−2, −2).
(e) Gradient $\frac{1}{2}$, through (8, 9).	(f) Gradient $-\frac{1}{2}$, through (−3, 0).
(g) Gradient $\frac{3}{2}$, through (9, 2).	(h) Gradient $-\frac{1}{3}$, through (7, −1).

11. Find the equation of each of the following straight lines.

(a) Through (2, 5) and (6, 9).	(b) Through (0, −1) and (2, −9).
(c) Through (14, 1) and (16, −5).	(d) Through (1, 1) and (2, 3).
(e) Through (1, 2) and (13, 6).	(f) Through (3, −2) and (−1, 6).
(g) Through (3, 9) and (0, 4).	(h) Through (0, 5) and (2, −5).

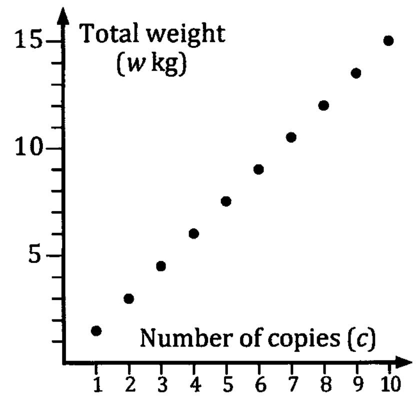
Linear relationships in practical situations.

Situation One at the start of this chapter involved copies of a particular book. Each copy weighed 1·5 kg so one book would weigh 1·5 kg, two books 3 kg, three books 4·5 kg, four books 6 kg and so on.

In this example the relationship between the total weight, w kg, and the number of copies, c, is exactly linear with all points exactly lying on the straight line

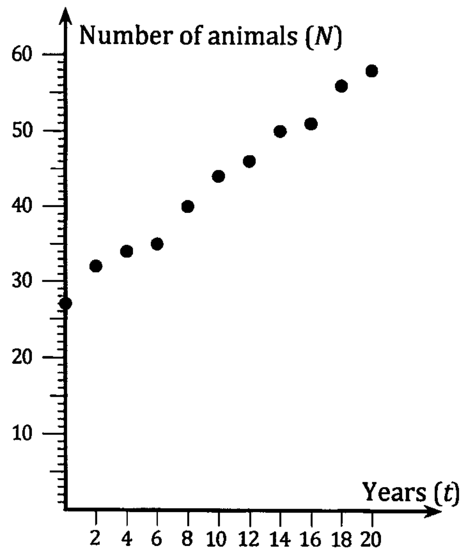
$$w = 1{\cdot}5\,c$$

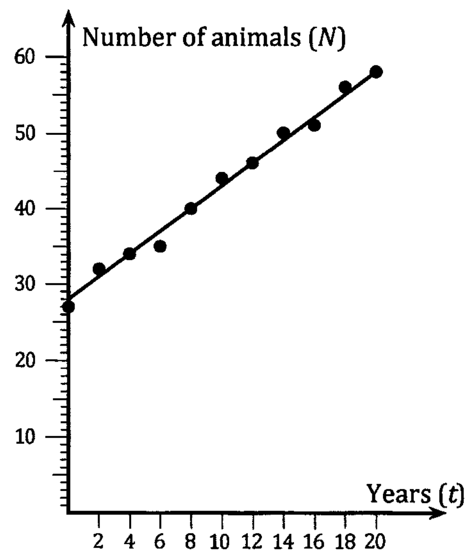
This is an example of a linear relationship in real life.



Some other situations may involve data that does not exactly fit a straight line but for which a straight line may be a reasonable *model* to use to summarise what is going on, and that might allow reasonable predictions and statements to be made.

Suppose, for example, that in the process of monitoring the survival of a particular endangered species of animal the numbers of these animals held in zoos around the world is recorded every two years over a period of twenty years. Suppose the numbers were as shown in the graph below left. Whilst the values do not exactly lie in a straight line, the rule $N = 28 + 1{\cdot}5t$ could be a reasonable linear model to use for this data, as shown below right.





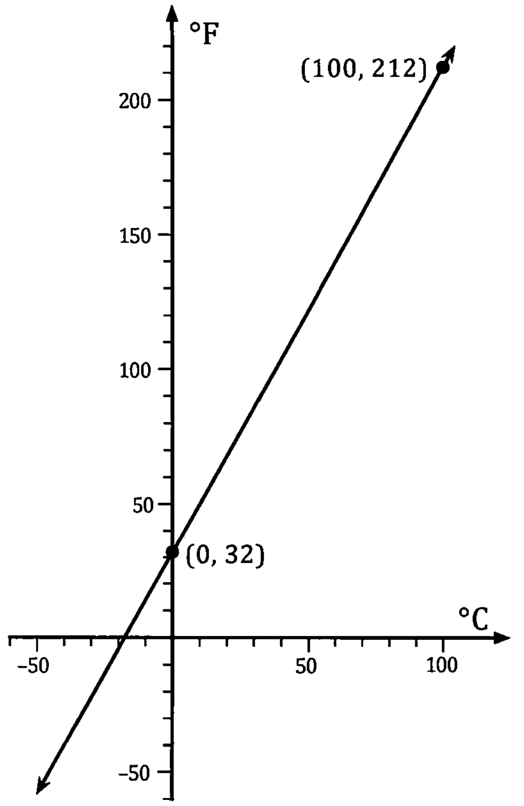
In this linear model ($N = 1{\cdot}5t + 28$):

☞ the vertical intercept is 28, indicating that the number of these animals kept in zoos at the beginning of the 20 year period was approximately 28,

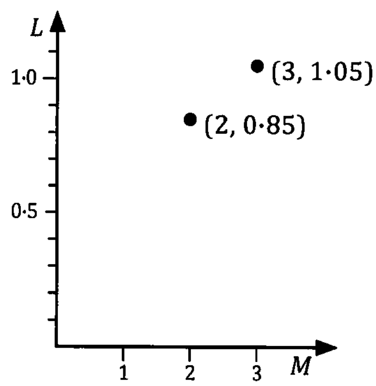
and ☞ the gradient is 1·5, indicating that on average each 1 year increase saw an increase of 1·5 in the number of these animals kept in zoos (i.e. an increase of approximately 3 every two years).

Exercise 8D

1. If we plot degrees Centigrade, (°C), on the x-axis and degrees Fahrenheit, (°F), on the y-axis, the graph for converting from one scale to the other is a straight line.
   (a) Given that 100°C is the same as 212°F and 0°C is the same as 32°F find the equation of the line in the form
   $$F = mC + b,$$
   where m and b are constants.
   (b) What does the value of m tell us about °C and °F temperatures?
   (c) Convert 55°C to °F.
   (d) Convert −10°C to °F.
   (e) Convert 59°F to °C.
   (f) Convert −4°F to °C.
   (g) Is there a temperature for which the number of degrees Centigrade is the same as the number of degrees Fahrenheit, and if so what is that temperature?

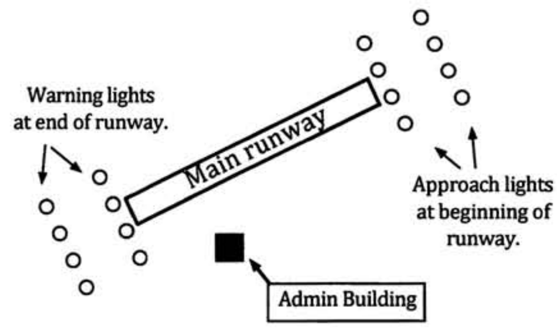


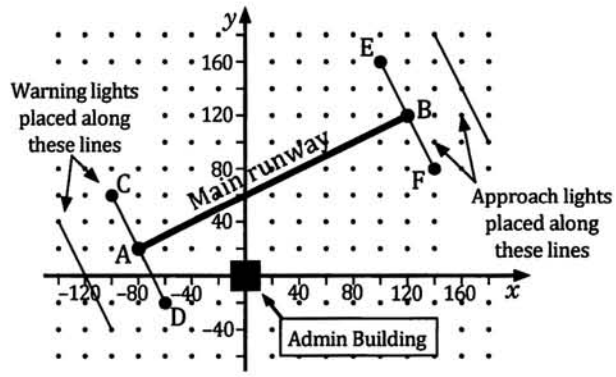
2. When a particular spring has a mass of M kg suspended from one end the total length of the spring is L metres where
   $L = \mathrm{k}M + L_0$ where k and L_0 are constants.
   (a) What will the value of k tell us about this situation?
   (b) What will the value of L_0 tell us about this situation?
   (c) A graph of M plotted on the x-axis and L on the y-axis passes through the points (2, 0·85) and (3, 1·05).



Calculate k and L_0 and hence determine how much the spring is extended **beyond its natural length** when a mass of 250 g is suspended from it.

3. The diagram on the right shows the proposed layout of a small airfield. The diagram shows the main runway, the approach lights, the warning lights and the administration building.
   The second diagram shows the proposal as a graph with lengths in metres and the admin building as the origin.
   Find:
   (a) the coordinates of the points A, B, C, D, E and F, (all divisible by 20).
   (b) the equation of the straight line through A and B,
   (c) the equation of the straight line through C and D,
   (d) the equation of the straight line through E and F.

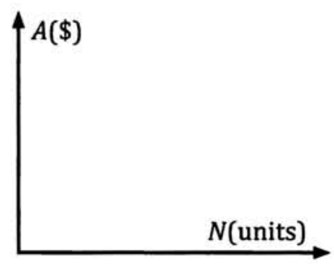


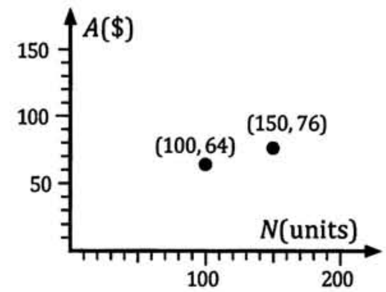


4. If we plot the "Number of metered units", N, on the x-axis and the "Amount to be paid", $\$A$, on the y-axis then the graph for calculating a telephone bill from one particular company is a straight line with equation

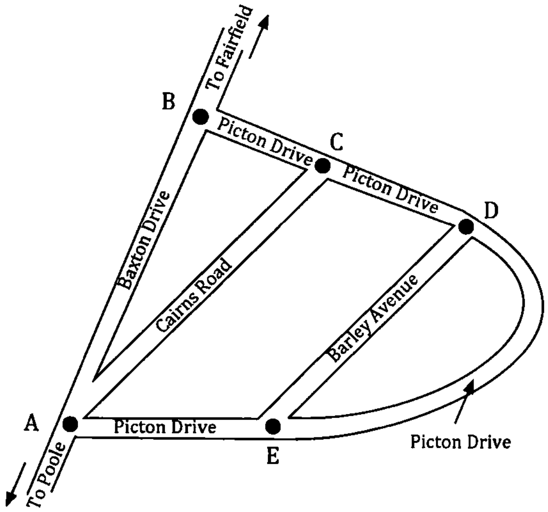
$$A = \mathrm{m}N + \mathrm{c},$$

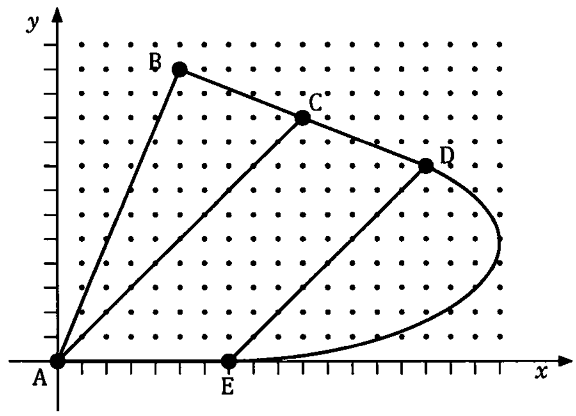
   where m and c are constants.
   In the context of this question:
   (a) what does the value of m tell us about A and N?
   (b) what does the value of c tell us?
   (c) If the bill for 100 units is \$64 and for 150 units is \$76, determine the equation of the line.
   (d) What would be the bill for 200 units?
   (e) If the bill was for \$82 how many units were used?





5. The diagram below left shows the proposed road system for a new housing estate off an existing road "Baxton Drive". The computer models this system graphically as shown below right, with 1 unit on each axis representing 25 metres.





Find the equation of the straight line through

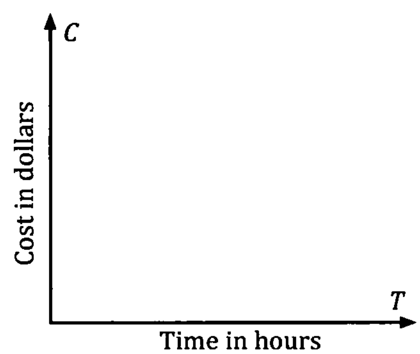
(a) A and B, (b) A and C, (c) A and E,

(d) E and D, (e) B, C and D.

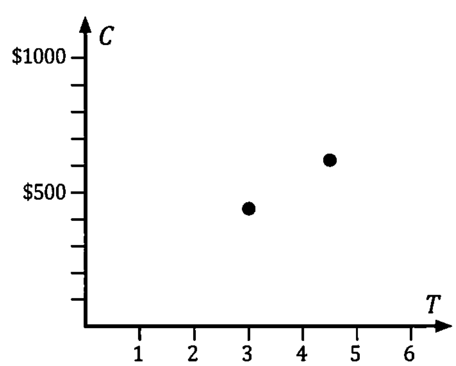
6. Susie Fuse, an electrician, charges her customers a standard call out fee plus a certain amount per hour. With this method of charging the cost to the customer, $\$C$, and the time taken to do the job, T hours, are linearly related and follow a rule of the form

$$C = \mathrm{m}T + \mathrm{c}.$$

Explain what information m, the gradient, and c, the intercept with the vertical axis, are giving in this context.



For a job that takes her three hours Susie charges $440 and for a job that takes her four and a half hours hours she charges $620.
Write the rule $C = \mathrm{m}T + \mathrm{c}$ with m and c evaluated.



7. A taxi company charges a fixed start fee (called the "flag fall" in the taxi industry) of \$4·90 followed by a charge of \$1·85 per kilometre.
If the cost for a journey of x kilometres is \$$C$, write a rule in the form

$$C = \mathrm{c} + \mathrm{m}x.$$

8. A water tank initially contains 1000 litres of water.
The tank develops a leak at its base, which causes water to leak out at a constant rate of 200 millilitres every minute.
If V litres is the volume of water in the tank t minutes after the leak commenced write a rule relating V and t.

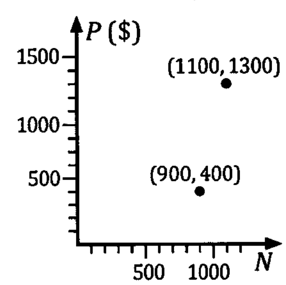
9. It costs a small business \$800 to run the business each week.
The company imports just one type of product from overseas and sells each one for \$75 more than it buys each one for.
Express \$$P$, the weekly profit the company makes, if it sells n of these products in that week.

10. A company sells copies of a book that it prints on a book by book basis, as orders arrive. Each book costs the company \$12 to produce in this way.
The company sells each book for \$22, which includes 10% goods and services tax.
The company forwards the goods and services tax amount to the government and the company keeps the rest.
If the company makes a profit of \$$P$ when it sells x copies of this book write a rule for P in the form $P = \mathrm{m}x$.

11. To cook a joint of meat a recipe book advises preheating the oven to 180°C and then, when the oven temperature has reached this temperature, place the meat in the oven for "20 minutes per kilogram + 20 minutes over".
Express as a rule the time, t hours, a joint of meat weighing k kilograms should be placed in the hot oven for it to cook according to these instructions.

12. A linear relationship exists between the profit, \$$P$, that the organisers of a concert make, and N, the number of tickets they sell. With P plotted on the vertical, y, axis and N on the horizontal, x, axis the line of this relationship passes through the points (900, 400) and (1100, 1300).
Find the equation of this line in the form

$P = \mathrm{m}N + \mathrm{c}$, where m and c are constants.



(a) What does the value of m tell us in this context?
(b) What does the value of c tell us in this context?
(c) What will be the profit when 1500 tickets are sold?
(d) If the concert hall has a maximum capacity of 2500 what profit will the organisers make if they give away 150 complimentary tickets and sell all the rest?
(e) What is the least number of tickets the organisers could sell and still not make a loss?

13. The owner of a computer shop calculates that his weekly profit from computer sales is linearly related to the number of computers sold that week.
If he sells 10 computers in a week his total profit is $560.
If he only sells 5 computers in the week he makes a profit of $10.
The rule relating his total profit for the week, $P, to the number of computers sold, x, is given by:

$$\text{Total profit in dollars } (P) = \text{m}x - \text{c},$$

$c being the fixed weekly cost of running the shop.
(a) Calculate m and c.
(b) What is his weekly profit from computer sales in a week that he sells 20 computers?

14. The membership secretary of a club monitors the growth in membership over a 5 year period. Plotting the 5 years (1, 2, 3, 4 and 5) on the horizontal "t" axis and the membership numbers on the vertical, "N", axis the secretary finds there is an almost perfect linear relationship between t and N. Express the relationship in the form $N = \text{m}t + \text{c}$ given that when $t = 1$, $N = 250$ and when $t = 5$, $N = 410$.
What do the values of the gradient, m, and the vertical intercept, c, tell us in the context of this question?
Use your equation to predict the value of N when $t = 10$, assuming the linear relationship continues.

15. The table below shows the profit, $P, that a company makes from the sale of x copies of a particular book it has had printed.

x	0	100	200	300	400	500	600	700	800
P	-3750	-2250	-750	750	2250	3750	5250	6750	8250

(a) Determine the rule for the relationship between P and x.
(b) How many copies of the book must the company sell to achieve a profit of more than $10 000?

16. The monitoring of the numbers of a particular endangered species of animal found that over a number of years, from the time the monitoring started, the numbers thought to be in existence in the wild showed a steady decline. Indeed with N representing the number of these animals thought to be in existence in the wild, t years into the monitoring program, N and t were approximately following the rule:

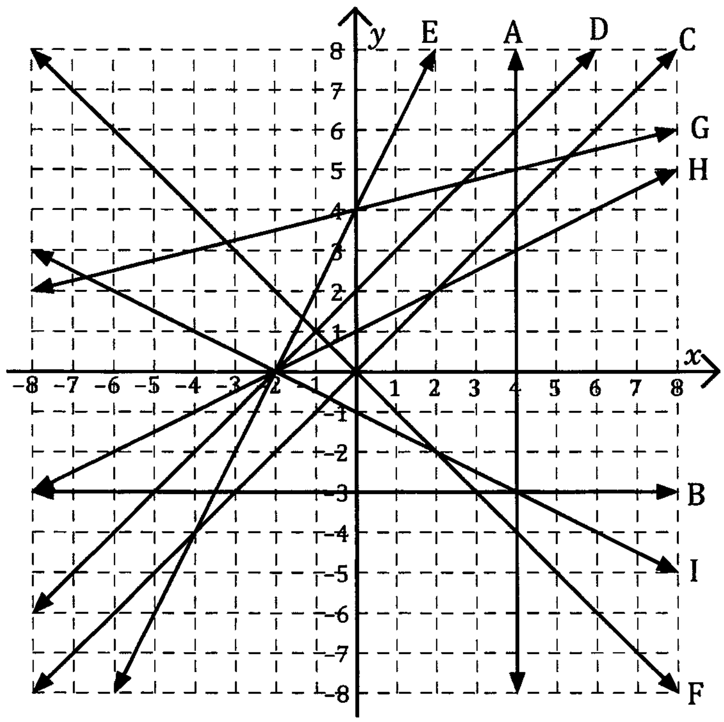
$$N = 5740 - 350t.$$

(a) Interpret what the numbers 5740 and 350 mean in the context of this situation.
(b) Graph the rule $N = 5740 - 350t$ with N plotted on the vertical axis and t on the horizontal axis.
(c) State the coordinates of the point where the line $N = 5740 - 350t$ cuts the horizontal axis and explain the significance of this point in the context of this question.

Miscellaneous Exercise Eight.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Determine the equation of each of the straight lines A to I shown below.



2. A particular straight line with a gradient of b and cutting the y-axis at the point with coordinates (0, a) has equation $y = a + bx$.
The line passes through the point (5, −2).
Which of the following equations must be true?

Equation 1	Equation 2	Equation 3
$5 = a - 2b$	$y = 5 - 2x$	$a + 5b + 2 = 0$

3. Find the values of $a, b, c, \ldots h$ in the following.

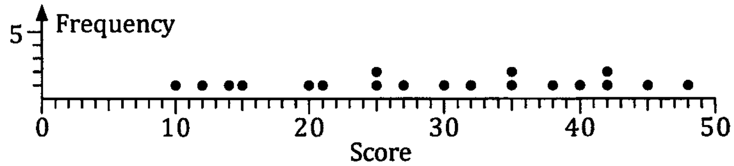
(a) $a : 2 = 6 : 5$	(b) $6 : 7 = 3 : b$	(c) $5 : 2 = 15 : c$	(d) $d : 2 = 19 : 10$
(e) $e : 3 = 9 : 2$	(f) $2 : 3 = 9 : f$	(g) $4 : g = 8 : 9$	(h) $4 : h = 9 : 8$

4. The ratio of females to males in a particular workforce is 7 : 9.
There are 252 males in this workforce.
How many females are there in this workforce?

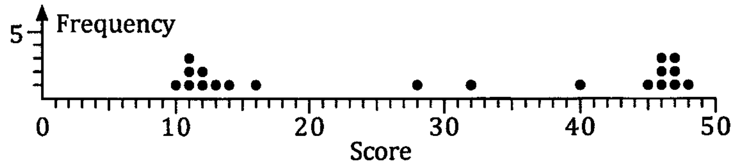
5. In the first three units of a course a student achieves a mean of 64%. In the next two units the student achieves a mean of 51%. What is the least mark the student needs to achieve in the sixth and final unit to gain an overall mean of at least 60%?

6. Four distributions of marks have both their dot frequency diagrams and their box plots shown below. Without the assistance of a calculator, match each dot frequency with its corresponding boxplot.

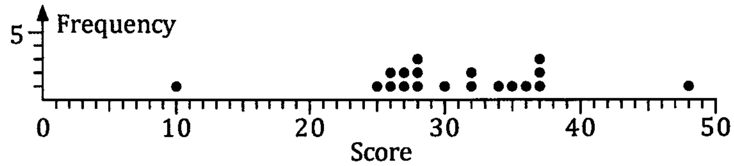
Dot frequency 1:



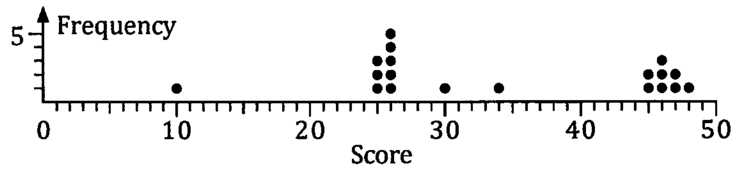
Dot frequency 2:



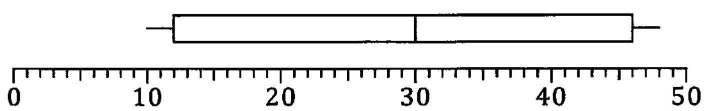
Dot frequency 3:



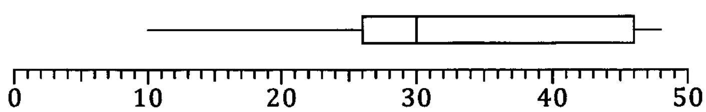
Dot frequency 4:



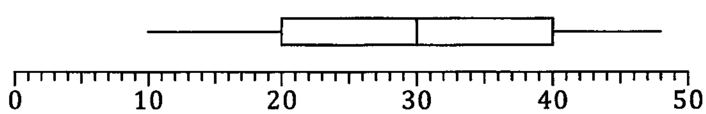
Box plot A:



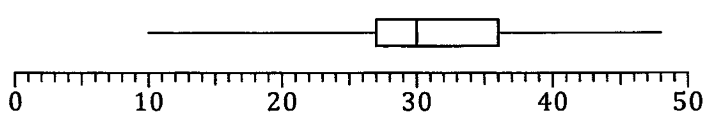
Box plot B:



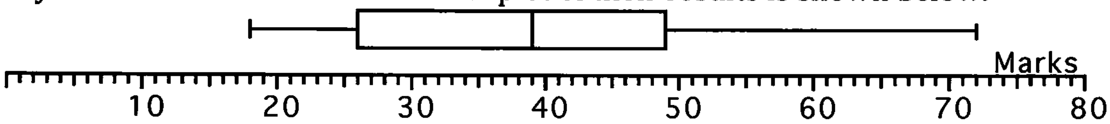
Box plot C:



Box plot D:

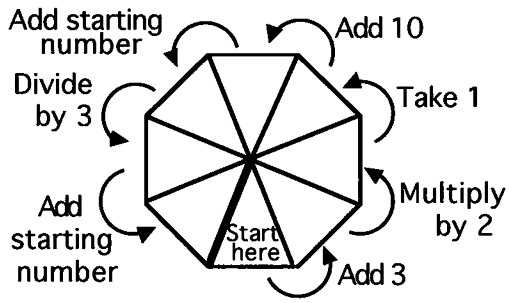


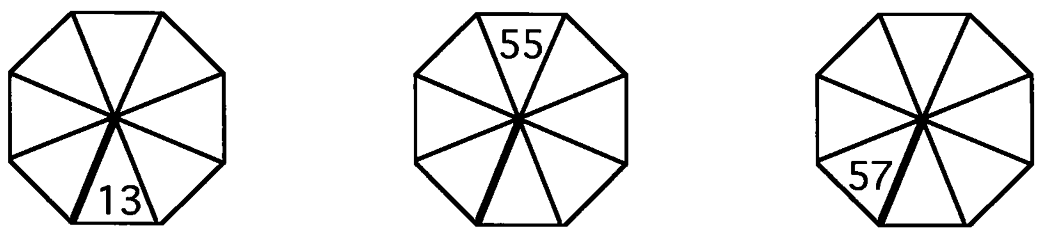
7. Forty one students sat a test. The boxplot of their results is shown below.



With the student who gained the top mark ranked 1st, the students ranked 19th and 21st in the test scored identical marks.
The top twelve students scored twelve different marks.
The student ranked 11th in the test scored 47.
(a) What was the top mark achieved in the test?
(b) What score did the student ranked 20th achieve?
(c) What score did the student ranked 10th achieve?

8. The three "octapatterns" below all follow the pattern shown on the right. Copy and complete each of the octapatterns shown below.





9. Certain types of cricket chirp more frequently as the temperature rises. For a particular species the number of chirps per minute (N) is found to be related to the Celsius temperature ($C°$) according to the rule:

$$N \approx 7C - 16$$

(a) If we were to graph $N = 7C - 16$, with C on the horizontal axis and N on the vertical axis it would give a straight line with gradient 7 and intersecting the vertical axis at −16. Interpret these two numbers in the context of this "chirping crickets" situation.
(b) What does the rule suggest as the temperature below which we would not expect a cricket to chirp?
(c) Roughly how many chirps will a cricket of this species make per minute if the temperature is (i) 14°C,
(ii) 28°C?
(d) Estimate the temperature if a cricket of this species is chirping
(i) 200 times per minute,
(ii) 50 times per 20 seconds.